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The Squares in Generalized Lucas Sequence

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ABSTRACT: In this paper we have defined generalized Lucas sequence $\left\{U_n^{(\alpha)}\right\}$ generalized companion Lucas sequence $\left\{V_n^{(\alpha)}\right\}$. The congruent properties proved for Lucas sequence $\{U_n\}$ and the Companion Lucas Sequence $\{V_n\}$. By using these properties the possible squares are identified in the generalized Lucas sequence $\left\{U_n^{(\alpha)}\right\}$ generalized companion Lucas sequence $\left\{V_n^{(\alpha)}\right\}$.

Key words: Square Number, Lucas Sequence, Companion Lucas Sequence, Generalized Lucas Sequence, Generalized Companion Lucas Sequence.

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I. INTRODUCTION

Suppose that P and Q be non-zero relatively prime integers. The Lucas sequence $\{U_n\}$ and the Companion Lucas Sequence $\{V_n\}$ with the parameters P and Q are defined by

$$U_0 = 0,\ U_1 = 1 \quad U_n = PU_{n-1} - QU_{n-2} \quad (n \ge 2) \qquad (1.1)$$

and $$V_0 = 2,\ V_1 = P,\ V_n = PV_{n-1} - QV_{n-2} \quad (n \ge 2) \qquad (1.2)$$

Paulo Ribenboim and Wayne L. Mc. Daniel [3] have proved that U_n is a square term only if $n = 0, 1, 2, 3$ *or* 6 and V_n is a square only if $n = 1, 3$ or 5 in these sequences. Later A. Bremner and N. Tzanakis [1] have proved that 12 th or 9th term is a square in the Lucas Sequence.

Now we define for a fixed integer $\alpha > 0$ two new sequences called Generalized Lucas Sequence $\left\{U_n^{(\alpha)}\right\}$ defined by

$$U_0^{(\alpha)} = 0,\ U_1^{(\alpha)} = 1$$

$$U_n^{(\alpha)} = \frac{\alpha}{2}(3\alpha - 1)U_{n-1}^{(\alpha)} - \frac{\alpha}{2}(\alpha + 1)U_{n-2}^{(\alpha)} \quad \text{for} \quad n \ge 2 \qquad (1.3)$$

and Generalized Companion Lucas Sequence $\left\{V_n^{(\alpha)}\right\}$ defined by

$$V_0^{(\alpha)} = 2, \quad V_1^{(\alpha)} = \frac{\alpha}{2}(3\alpha - 1)$$

$$V_n^{(\alpha)} = \frac{\alpha}{2}(3\alpha - 1)V_{n-1}^{(\alpha)} - \frac{\alpha}{2}(\alpha + 1)V_{n-2}^{(\alpha)} \quad \text{for} \quad n \ge 2. \qquad (1.4)$$

We have proved possible squares in the sequences (1.3) and (1.4).

II. The first few polynomials of Generalized Lucas Sequence $\left\{U_n^{(\alpha)}\right\}$ and Generalized Companion Lucas Sequence $\left\{V_n^{(\alpha)}\right\}$ are given as follows.

Table 1(a) Generalized Lucas Sequence.

n	$U_n^{(\alpha)}$
0	0
1	1
2	$\frac{1}{2}(3\alpha^2 - \alpha)$
3	$\frac{1}{4}(9\alpha^4 - 6\alpha^3 + \alpha^2 - 2\alpha)$
4	
5	$\frac{1}{8}(27\alpha^6 - 27\alpha^5 - 3\alpha^4 - 9\alpha^3 + 4\alpha^2)$
	$\frac{1}{16}(81\alpha^8 - 108\alpha^7 - 30\alpha^5 + 35\alpha^4 + 2\alpha^3 + 4\alpha^2)$

Table 1(b) Generalized Companion Lucas Sequence.

n	$V_n^{(\alpha)}$
0	2
1	$\frac{1}{2}(3\alpha^2-\alpha)$
2	$\frac{1}{4}(9\alpha^4-6\alpha^3-3\alpha^2-4\alpha)$
3	$\frac{1}{8}(27\alpha^6-27\alpha^5-9\alpha^4-13\alpha^3+6\alpha^2)$
4	$\frac{1}{16}(81\alpha^8-108\alpha^7-18\alpha^6-36\alpha^5+49\alpha^4+8\alpha^3+8\alpha^2)$
5	$\frac{1}{32}(243\alpha^{10}-405\alpha^9-90\alpha^7+255\alpha^6+195\alpha^5+30\alpha^4-20\alpha^3)$

III. In this section we have presented some properties of $\{U_n\}$ and $\{V_n\}$. The following properties are well known, in fact for all integers m and n.

$$U_n=\frac{a^n-b^n}{a-b} \text{ and } V_n=a^n+b^n \tag{3.1}$$

Where $a=\frac{1+\sqrt{5}}{2}$ and $b=\frac{1+\sqrt{5}}{2}$

$$U_{m+n}=U_mV_n-a^nb^nU_{m-n} \tag{3.2}$$
$$V_{m+n}=V_mV_n-a^nb^n\ V_{m-n} \tag{3.3}$$
$$U_{2n}=U_nV_n \tag{3.4}$$
$$U_{2n+1}=U_{n+1}V_n-(ab)^n \tag{3.5}$$
$$V_{2n}={V_n}^2-2(ab)^n \tag{3.6}$$
$$V_{2n+1}=V_{n+1}V_n-(a+b)(ab)^n \tag{3.7}$$

3.8 The congruent properties of $\{U_n\}$ and $\{V_n\}$ are proved as follows.

Lemma: (i) $U_{2k}\equiv -1\ (mod\ 8)$ for $k\geq 2$

(ii) $V_{2k}\equiv -1\ (mod\ 8)$ for $k\geq 2$

(iii) $U_{2k+1}\equiv -1\ (mod\ 8)$ for $k\geq 2$

(iv) $V_{2k+1}\equiv 1\ (mod\ 8)$ for $k\geq 2$

Proof: (i) is trivial if $k=2$, since $U_4=-1$ from table 2(C)

For $k\geq 2$, we have by (3.4) $U_{2k}=U_kV_k$

$U_{2k}=U_2V_2=1(-1)\quad\equiv -1\ (mod\ 8)$

(ii) by (3.6), we have $V_{2k}={V_k}^2-2(ab)^k$

For $k\geq 2$, $V_{2k}={V_2}^2-2.1\ =(-1)^2-2\equiv -1\ (mod\ 8)$

(iii) by (3.5), we have $U_{2k+1}=U_{k+1}V_k-(ab)^k$

For $k=2$, $U_{2k+1}=U_3.V_2-(ab)^2\quad\equiv -1\ (mod\ 8)$

(iv) by (3.7), we have $V_{2k+1}=V_{k+1}V_k-(a+b)(ab)^k$

For $k=2$, $V_{2k+1}=V_3V_2-\ (a+b)(ab)^2=(-2)(-1)-1\equiv 1\ (mod\ 8)$

Which completes the proof of the Lemma.

3.9. Note: For any integer m, we have

(i) $m^2\equiv 0,1 or\ 4\ (mod\ 8)$

(ii) $m^2\equiv 0,1,4,9,16,17\ or\ 25\ (mod\ 32)$

IV. MAIN THEOREMS

4.1 Theorem: $U_n^{(1)}$ is a square if and only if $n=1\ or\ 2$.

Proof: Let $U_n^{(1)}=U_n$, clearly U_1 and U_2 are squares.

Conversely, suppose $U_n=m^2$ for some integer m. Then $n=1\ or\ 2$, since for $n>2$, we have $m^2=U_n\equiv -1\ (mod\ 8)$. By (i) and (iii) of Lemma, which cannot true for the note (3.7).

4.2 Theorem: $V_n^{(1)}$ is a square if and only if $n=1$.

Proof: Let $V_n^{(1)}=V_n$, clearly V_1 is a square.

Conversely, suppose $V_n=m^2$ for some integer m. Let neither n is even nor an odd integer greater than 1. In fact, if $n=2k$ where $k\geq 0$, $V_n=a^n+b^n=2$. Which is obviously not a square. If $n=2k+1\ and\ k\geq 1$ then (iv) of Lemma gives $V_n\equiv 1\ (mod\ 8)$. Proving that V_n cannot be a square.

4.3 Theorem: $U_n^{(\alpha)}$ is a square if $n=0\ or 1$.

Proof: We prove the theorem by the principle of mathematical induction on α.

We have $U_0^{(\alpha)} = 0$ and $U_0^{(\alpha)} = 1$, Put $\alpha = 1$, from theorem 4.1 and $U_0^{(1)}$ and $U_1^{(1)}$ are squares. Therefore the result is true for $\alpha = 1$. Assume that it is true for $\alpha = m$. We have to prove it is true for $\alpha = m + 1$, then $U_0^{(m+1)} = 0$ *and* $U_1^{(m+1)} = 1$ (for all α).

4.4 Theorem: $U_n^{(1)} + 1$ is a square if and only if $n = 0$ *or* 3.

Proof: Let $U_n^{(1)} = U_n$, clearly $U_0 + 1 = 1$ *and* $U_3 + 1 = 1$, Part of the theorem holds.

Conversely, suppose $U_n + 1$ is a square, then n cannot be even and $n > 0$, since $U_2 + 1 = 2$ and for $m \geq 2$, we have $U_{2m+1} + 1 \equiv 0 \pmod{8}$, by Lemma 3.7. Again if $n = 2k + 1$ with $k \geq 2$ then $U_n + 1 \equiv 12 \pmod{32}$, Therefore $U_n + 1$ is not a square by note(3.9)

4.5 Theorem: $U_n^{(\alpha)} + 1$ is a square if $n = 0$.

Proof: We prove the theorem by the principle of mathematical induction on α.

We have $U_0^{(\alpha)} = 0$. Put $\alpha = 1$, from theorem 4.4 and $U_0^{(\alpha)} + 1 = 0 + 1 = 1$. Therefore the result is true for $\alpha = 1$. Assume that the result is true for $\alpha = m$. We have to prove that it is true for $\alpha = m + 1$. Put $\alpha = m + 1$, then $U_0^{(m+1)} = 0 + 1 = 1$ (for all α). By the principle of mathematical induction $U_n^{(\alpha)} + 1$ is a square for all integers $\alpha > 0$.

4.6 Theorem: $V_n^{(1)} - 1$ is a square if and only if $n = 0$.

Proof: Let $V_n^{(1)} = V_n$, if $n = 0$, clearly $V_0 - 1 = 1$

Conversely, suppose $V_n - 1$ is a square, if $n > 1$ cannot hold. Then $n = 2m + 1$ for some integer $m \geq 1$ by Lemma 3.8, we have $V_n - 1 = V_{2m+1} - 1 \equiv 1 \pmod{8}$ and hence $V_n - 1$ is not a square.

4.7 Theorem: $V_n^{(\alpha)} - 1$ is a square if $n = 0$.

Proof: We prove the theorem by the principle of mathematical induction on α.

We have $V_0^{(\alpha)} = 2$. Put $\alpha = 1$, from theorem 4.6 and we have $V_n^{(1)} - 1 = 1$ is a square if $n = 0$. Therefore the result is true for $\alpha = 1$. Assume that the result is true for $\alpha = m$. We have to prove that it is true for $\alpha = m + 1$. Then $\alpha = m + 1$, then $V_0^{(m+1)} - 1 = 1$ *(for all* α*)*. Therefore by the principle of mathematical induction $V_n^{(\alpha)} - 1$ is a square for all integers $\alpha > 0$ and for $n = 0$.

V. DISCUSSIONS

The recursive sequences (1.3) and (1.4) the following are always square numbers (i) $U_n^{(1)} - 1$ is never a square for $n \leq 5$

(ii) $V_n^{(1)} + 1$ is never a square for $n \leq 5$

(iii) $U_n^{(5)} + 1$ is a square if $n = 2$

(iv) $V_n^{(5)} + 1$ is a square for $n = 1$

(v) $V_n^{(2)} + 1$ is a square for $n = 3$

VI. CONCLUSION

The Lucas sequence $U_n^{(\alpha)}$ is a square if $n = 1$ *or* 2 and $U_n^{(1)} + 1$ is a square if and only if $n = 0$ *or* 3. The companion Lucas sequence $V_n^{(\alpha)}$ is a square for $n = 1$.

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